Dear Mathematics Teachers:

The arithmetical example that follows illustrates a method I devised for arriving at an approximation to the square root of any real number. I have not been able to find this method in the literature. Perhaps it will be of some use to you when you teach square roots.

The method recommends itself as to its simplicity, transparency, and ease with which the calculations may be carried out by any person with a mastery of addition, subtraction, multiplication and division.

I published an article on this method entitled “Beating the square root out of a radical sign with the number one” at the *Op Ed News* website (www.opednews.com) website, and a slightly different version on my own website, Peter’s New York (www.petersnewyork.com) in June of 2011.

Thank you for taking the time to review this material. Any feedback you may have will be very much appreciated.

Regards,

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Example: finding the square root of 71.

The nearest integral perfect square less than 71 is 64, the square root of which is 8. The next perfect square is 81, the square root of which is 9. As we want to approach the solution of our problem from lower values, we use 64.

$\sqrt{71}$
$= \sqrt{[71 \times 64/64]}$
$= \sqrt{[64 \times (71/64)]}$
$= \sqrt{64} \times \sqrt{(71/64)}$
$= 8 \times \sqrt{(71/64)}$

Now we perform the division 71/64 to put the fraction into decimal form.

```
     1.109375
64 ⌈ 71.000000000
    64
     70
     64
      600
      576
       240
       192
        480
        448
         320
         320
           0
```

Now viewing the result of the division, we can quickly reach a conclusion as to the number which when squared gives us a close approximation to this result. I can say it is 1.05. Why? Because, as it turns out, when we multiply this number by itself, the .05's are going to be added to one another, to yield the 1.10. And through this convenient property, which recurs throughout our calculations, we can very quickly estimate the number which when squared will yield the number we want. It is generally one plus half of the first figure following the decimal point. (For example, if I have $\sqrt{1.0832}$, my quick guess for the estimated square root is 1.04. However, if it were $\sqrt{1.080003}$, we would find that 1.04 x 1.04 is too great, or 1.0816. So we would go down to 1.03, which when squared would give us 1.0609.)

```
    1.05
    1.05
    5 25
 1 05
 1.10 25
```

which is less than 1.109375. As mentioned, this is very often, but not always the case. If for some reason, the number had exceeded the number required, we could have gone down a hundredth, to 1.04, and tried again.

We now rewrite our expression:

= 6 x √[1.1025 x (1.109375/1.1025)]

= 6 x 1.05 x √(1.109375/1.1025)

We have indeed made some progress. But to ensure the accuracy of our result, it is best to carry out our program for at least three or four cycles. So we perform the required division:

1.1025 ⎾ 1.109375

```
                 1.0062358276
= 11025 ⎾ 11093.7500000000000
          11025
             68750
             66150
              26000
              22050
               39500
               33075
                64250
                55125
                 91250
                 88200
                  30500
                  22050
                   84500
                   77175
                    73250...
```

where we stop our division, perceiving that we have carried it out to enough decimal places to ensure the accuracy of our results to the degree desired. I caution against rounding upward, as it will yield an expression greater than the one we seek.

The results of our division immediately suggest the number we search for, the square of which will be less than, but will closely approximate, the dividend. It is simply 1.003.

We carry out the desired multiplication:

```
      1.003
      1.003
      3 009
  1 003
  1.006 009
```

We once again rewrite our expression according to the progress thus made:

= 8 x 1.05 x √[1.006009 x (1.0062358276/1.006009)]

= 8 x 1.05 x 1.003 x √(1.0062358276/1.006009)

Now we again carry out the required division:

1.006009 ⌈1.0062358276

```
                     1.000225
=  1006009 ⌈1006235.827600
             1006009
                 2268276
                 2012018
                  2562580
                  2012018
                   5505620
```

That is enough for our purposes, because we can immediately discern the number whose square will yield approximately our dividend. It is, of course, 1.00011:

1.00011
1.00011
1 00011
10 0011
1.000**11**
1.000**22** 00122

We have thus ended our calculations, and produce a series of products that yield the approximate result desired, this series being: 8 x 1.05 x 1.003 x 1.00011.

And as we have always been careful to underestimate, we shall arrive at a figure whose square will also understate, by some small quantity, the square root of 71.

Let's look at our products as we carry out each multiplication.

We started with 8, while 9 was seen as too great.

8 x 8 = 64; 9 x 9 = 81, thus, the square root of 71 lies between 8 and 9.

We can do likewise with the other estimates to gauge the range within which our square root must be found.

8	< $\sqrt{71}$ <	9
1.0**5**		1.0**6**
8		8
8.40	< $\sqrt{71}$ <	8.48
8.40		8.40
1.0 0**3**		1.0 0**4**
2 5 20		3 3 60
8 40		8 40
8.42 5 20	< $\sqrt{71}$ <	8.43 3 60
8.42 5 20		8.42 5 20
1.00 0 **11**		1.00 0 **12**
8 42 5 20		16 85 0 40
84 25 2 0		84 25 2 0
8 42520		8 42520
8.42612 67 7 20	< $\sqrt{71}$ <	8.42621 10 2 40

The proof, as is often said, is in the pudding, so let's square those digits that are in agreement between the final higher and lower estimate:

```
       8.426
       8.426
      50 556
     168 52
    3 370 4
  67 408
  70.997 476
```

approx. = 71, the number whose square root was to be found. Our upper and lower estimates, taken to two more decimal places, and using a calculator, to conserve ink or pencil lead, yields the following:

70.9994982544 < 71 < 71.0010149641

We could have enhanced our accuracy by continuing to carry out the appropriate operations.

In sum, our method is to multiply and divide the number under the radical sign by a perfect square near in value to the number, but not exceeding it. We then merely take the perfect square out of the radical sign, where it becomes its square root, which we already know, while within the radical sign, we divide the original number by the perfect square, yielding a new number upon which we merely repeat the process as many times and in any way we like.

One can use a calculator to demonstrate this method, with much faster results. It can be used for numbers of any magnitude. For example, √738591 can be rewritten as a product of a decimal and ten to an even power: √[**73.8591 x 10^4**] = 10^2 x √73.8591. We then begin: 8 x 8 = 64, thus:

10^2 x 8 x √[73.8591/64] = 10^2 x 8 x √1.1540484375 = 10^2 x 8 x 1.05 x √1.1540484375/1.1025 = 10^2 x 8 x 1.05 x √1.0467559523809523809523809523 81 = 10^2 x 8 x 1.05 x 1.02 x √1.046755952380952380952380952381 /1.0404 = √1.006109143003606671426740630 8929 = 10^2 x 8 x 1.05 x 1.02 x 1.003 √1.00610914300.../1.006009 = 10^2 x 8 x 1.05 x 1.02 x 1.003 x √1.000099 = approximately 10^2 x 8 x 1.05 x 1.02 x 1.003 x 1.00005 (I took the higher value for the last "estimate,"100005, which yields a result slightly greater than the one we seek) = 10^2 8.5941336852 which when squared gives us **10^4 x 73.8591**3379908933269904 (I have used an electronic device to facilitate calculations in this last example). I suspect that cube roots and higher order roots could also be obtained using this same method.
QED